YEAR 13 MATHEMATICS WEEK 8 HOMEKIT

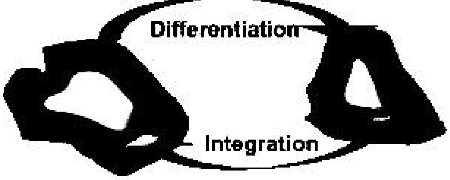
STRAND NINE

INTEGRATION

SUB – STRAND 9.1

Anti - derivatives of Functions

$$\frac{d(x^3)}{dx} = 3x^2$$



$$\int 3x^2\,dx = x^3 + c$$

FUNDAMENTAL THEOREM OF CALCULUS

- *The fundamental theorem of calculus states that differentiation and integration are inverse operations. More precisely, it relates the values of antiderivatives to definite integrals. Because it is usually easier to compute an antiderivative than to apply the definition of a definite integral, the fundamental theorem of calculus provides a practical way of computing definite integrals. It can also be interpreted as a precise statement of the fact that differentiation is the inverse of integration.*

LESSON 60

LEARNING OUTCOMES

9.1.1 Evaluate integrals using u-substitution.

9.1.1 Algebraic Substitution

9.1.1.1 Type I and Type II

- Integration by u Substitution: an u substitution transforms the given integrals into easier ones.

Steps:

1. Make an appropriate choice for u. Usually we take u to be an expression whose derivative appears as a factor of the integrand.
2. Compute $\frac{du}{dx}$.
3. Make dx the subject.
4. Substitute u and dx from 3 . Check that the integral is now in terms of u. This new integral should be easier than the initial one. If not, then your u Substitution is incorrect. Go back to step 2 and come up with another substitution.
5. Evaluate the resulting integral.
6. Do not forget that the answer is a function of x. You should substitute back the initial **variable** x.

For some special cases (Type II), there is a need to convert the integrand to an expression that can be easily integrated.

All the steps are similar to u Substitution but **with one change**:

- Need to make x the subject from the "u" equation

Also, note if $f(x) = x^n$, $\int f(x)\, dx = \boxed{\int x^n\, dx = \frac{x^{n+1}}{n+1} + C}$

The *definite integral* of *f(x)* between *a* (lower limit) and *b* (upper limit) can be defined as follows:

$$\int_a^b f(x)dx = [\mathrm{F}(x)]_a^b$$
$$= \mathrm{F}(b) - \mathrm{F}(a)$$

Example 1 Find $\int \frac{2x}{x^2-3}\,dx$

Answer: Type I Substitution

Take u to be the denominator since its derivative appears in the numerator.

Let $u = x^2 - 3$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$\frac{du}{2x} = dx$$

Write integral in terms of u.

$$\int \frac{2x}{x^2+3}\,dx = \int \frac{2x}{u}\,\frac{du}{2x}$$

$$= \int \frac{1}{u}\,du$$

Evaluate the resulting integral

Substitute $u = x^2 + 3$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$\therefore \int \frac{2x}{x^2-3}\,dx = \ln\left|x^2-3\right| + C$$

Short cut: If the derivative of the denominator = numerator then take the *ln* of the absolute value of denominator.

For example $\int \frac{3x^2}{x^3-2}\,dx = \ln\left|x^3-2\right| + C$

Example 2 Find $\int \frac{x^2}{\sqrt{x^3+9}}\,dx$

Answer: Type I Substitution

Take u to be x^3+9 since its derivative appears in the numerator.

$$u = x^3 + 9$$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx \quad \Rightarrow \frac{du}{3x^2} = dx$$

Write integral in terms of u.

$$\int \frac{x^2}{\sqrt{x^3+9}}\,dx \quad = \int \frac{x^2}{\sqrt{u}} \frac{du}{3x^2}$$

$$= \int \frac{du}{3\sqrt{u}}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$$

Evaluate the resulting integral.

$$\frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{3}\int u^{-\frac{1}{2}} \quad du$$

Simplify

$$= \frac{1}{3}\left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = \frac{1}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}\sqrt{u} + C$$

substitute for u

$$\therefore \int \frac{x^2}{\sqrt{x^3+9}} \quad dx = \frac{2}{3}\sqrt{x^3+9} + C$$

Example 3 Find $\int x(3x+1)^9\,dx$

Answer: Type II Substitution

Take u to be the expression in the brackets.

$$u = 3x+1$$

$$\frac{du}{dx} = 3$$

$$\frac{du}{3} = dx$$

Make "dx" the subject.

Substitute

$$\int x(3x+1)^9\,dx = \int xu^9 \frac{du}{3}$$

$$= \frac{1}{3}\int xu^9\,du$$

In this case x do not cancel as in examples 1 and 2.

To remove x, make x the subject from the "u" equation

$$x = \frac{u-1}{3}$$

Substitute x to write integral in terms of u.

Simplify

$$\int x(3x+1)^9.dx = \int \frac{u-1}{3}.u^9 \frac{1}{3} du$$

$$= \int \frac{u^{10} - u^9}{9} du$$

$$= \frac{1}{9}\int u^{10} - u^9.du$$

Evaluate the resulting integral

$$\frac{1}{9}\int u^{10} - u^9.du = \frac{1}{9}\left(\frac{u^{11}}{11} - \frac{u^{10}}{10}\right) + C$$

$$= \frac{u^{11}}{99} - \frac{u^{10}}{90} + C$$

Substitute for u

$$\therefore \int x(3x+1)^9\, dx = \frac{(3x+1)^{11}}{99} - \frac{(3x+1)^{10}}{90} + C$$

Example 4 Evaluate $\int_0^2 x(x^2+1)^3 dx$

Answer

Let $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$\Rightarrow \frac{du}{2x} = dx$$

$$\int x(x^2+1)^3\, dx = \int x\, u^3 \frac{du}{2x}$$

$$= \int \frac{u^3}{2} du$$

$$= \frac{u^4}{8} + C$$

Method 1: Find the limits of integration in terms of u

When $x = 0$	When $x = 2$
$u = x^2 + 1$	$u = x^2 + 1$
$u = 0^2 + 1$	$u = 2^2 + 1$
$= 1$	$= 5$

$$\left[\frac{u^4}{8}\right]_1^5 = \frac{5^4}{8} - \frac{1^4}{8}$$

$$= 78$$

Method 2: Evaluate in terms of x

$$\frac{u^4}{8} = \frac{(x^2+1)^4}{8}$$

$$\left[\frac{(x^2+1)^4}{8}\right]_0^2 = \frac{(2^2+1)^4}{8} - \frac{(0^2+1)^4}{8}$$

$$= 78$$

<u>Class Activity 60</u>

1. Evaluate the following:

a) $\int \frac{3}{2x+1} dx$ b) $\int_2^{10} \frac{1}{x+2} dx$

c) $\int \frac{3x}{x^2 - 2} dx$ d) $\int \frac{3x^2 + 2}{x^3 + 2x} dx$

2. Evaluate the following:

a) $\int_1^9 x(x^2+1)^5 \, dx$

b) $\int \frac{x}{\sqrt{5+x}} dx$

LESSON 61: **Trigonometric, Exponential and Hyperbolic Function**

Learning Outcome: Evaluate integrals using u-substitution.

- The derivative and integration are opposite process.

- Hyperbola

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

- The exponential function

$$\int e^{x}\, dx = e^{x} + C \Rightarrow \int e^{ax}\, dx = \frac{e^{ax}}{a} + C$$

- Trigonometric function

$\int \cos x\, dx = \sin x + C$
$\int -\sin x\, dx = \cos x + C$ $\int \sin x = -\cos x + C$
$\int \sec^{2} x = \tan x + C$

Example 1 Find $\int \frac{1}{5x} dx$

Answer

$$\int \frac{1}{5}\left(\frac{1}{x}\right)\, dx = \frac{1}{5}\int \frac{1}{x} dx$$

$$= \frac{1}{5}\ln|x| + C$$

Example 2 Find $\int \sin 3x \, dx$

Answer

$$u=3x, \quad \frac{du}{dx}=3, \quad \frac{du}{3}=dx$$

$$\int \sin 3x \, dx = \int \sin u \, \frac{du}{3}$$

$$= \frac{1}{3}\int \sin u \, du$$

$$= \frac{1}{3}(-\cos u)+C$$

Substitute for u

$$\therefore \int \sin 3x \, dx = \frac{-\cos 3x}{3} + C \quad or \quad -\frac{1}{3}\cos 3x + C$$

Short cut: integrate the outer function (evaluated at inner) divide by the derivative of inner linear function

Example 3 Find $\int e^{2x+1} \, dx$

Answer

Method 1: Integration by u Substitution

Take u to be the power (exponent)

Let $u = 2x + 1$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{du}{2} = dx$$

Write integral in terms of u.

$$\int e^{2x+1} dx = \int e^{u} \frac{du}{2}$$

Evaluate the resulting integral.

$$\frac{1}{2}\int e^{u} \, du = \frac{1}{2}e^{u} + C$$

Substitute for u

$$\therefore \int e^{2x+1} dx = \frac{e^{2x+1}}{2} + C$$

Method 2: integrate the outer function (evaluated at inner) divide by the derivative of inner linear function.

$f(x) = 2x + 1, \quad f'(x) = 2$

$$\therefore \int e^{f(x)} \, dx = \frac{e^{f(x)}}{f'(x)} + C$$

$$= \frac{e^{2x-1}}{2} + C$$

Example 4 Evaluate $\int_{1}^{e+1} \frac{1}{x-1} dx$

Answer

$$\int_{1}^{e+1} \frac{1}{x-1} dx = \Big[\ln|x-1| \Big]_{1}^{e+1}$$

$$= \ln|e+1-1| - \ln|1|$$

$$= \ln e - \ln 1$$

$$= 1$$

Short cut: If the derivative of the denominator = numerator then take the *ln* of the absolute value of denominator.

Example 5 Find $\int \cos x . e^{\sin x} . dx$

Answer

Take u to be the power.

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow \frac{du}{\cos x} = dx$$

Write integral in terms of u.

$$\int \cos x . e^{\sin x} . dx = \int \cos x \; e^{u} \; \frac{du}{\cos x}$$

$$= \int e^{u} du$$

Evaluate the resulting integral.

$$\int e^{u} du = e^{u} + C$$

Substitute for u

$$\therefore \int \cos x . e^{\sin x} dx = e^{\sin x} + C$$

Example 6 Evaluate $\int_{0}^{\ln 2} 3e^{x} \, dx$

Answer

$$\int_{0}^{\ln 2} 3e^{x} \, dx = \Big[3e^{x} \Big]_{0}^{\ln 2}$$

$$= 3e^{\ln 2} - 3e^{0}$$

$$= 3 \times 2 - 3$$

$$= 3$$

Example 7 Evaluate $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$

Answer

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow \frac{du}{\cos x} = dx$$

$$\int \cos x \sin^2 x . dx = \int \cos x \; u^2 \; \frac{du}{\cos x}$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

Method 1: Find the limits of integration in terms of u

When $x = 0$	When $x = \frac{\pi}{2}$
$u = \sin x$	$u = \sin x \Rightarrow u = \sin \frac{\pi}{2}$
$u = \sin 0$	$= 1$
$= 0$	

$$\left[\frac{u^3}{3}\right]_0^1 = \frac{1^3}{3} - 0$$

$$= \frac{1}{3}$$

Method 2: Evaluate in terms of x

$$\frac{u^3}{3} = \frac{\sin^3 x}{3}$$

$$\left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{\sin^3\left(\frac{\pi}{2}\right)}{3} - \frac{\sin^3 0}{3}$$

$$= \frac{1}{3}$$

Class Activity 61

1. Evaluate the following:

$a)\ \int \sin 5x\, dx$ $\qquad$ $b)\ \int e^{-2x} dx$

2. Find the anti-derivatives of the following:

$a)\ \int xe^{x^2}\, dx$

b) $\int \frac{e^x}{e^x + 1} dx$

3. Evaluate the following:

$a)\ \int_0^{\pi} \cos(2x)\ dx$ $\qquad$ $b)\ \int_0^1 e^{5x}\ dx$ $\qquad$ $c)\ \int_1^3 1 + \frac{1}{x}\ dx$

SUB – STRAND 9.2

Applications of Integration

Indefinite Integral

General: $\int f(x)dx = F(x) + C$

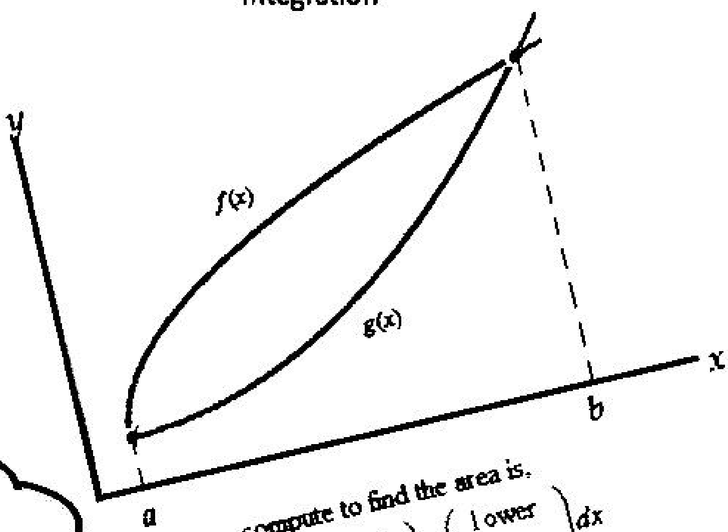
This represents a *family* of functions

- Integral Sign
- Integrand
- Variable of Integration
- The Indefinite Integral
- The constant of Integration

definite Integral

This gives you a *number*

Definite: $\int_a^b f(x)dx = F(b) - F(a)$



compute to find the area is,

$$A = \int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right) dx$$

LESSON 62

LEARNING OUTCOMES

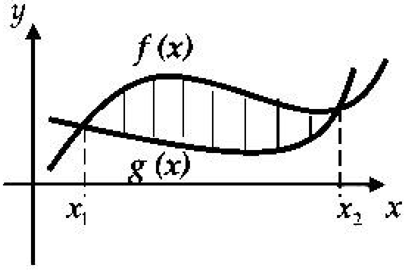
9.2.1 Calculate the area between two graphs.

9.2.1 Area Between Two Graphs

If the upper curve or line on the top is $f(x)$ and the lower curve or line at the bottom is $g(x)$, then the area between two curves or lines from a to b is given by:

$$Area = \int_{Lower\ Limit}^{Upper\ Limit} (Upper\ Curve - Lower\ Curve)$$

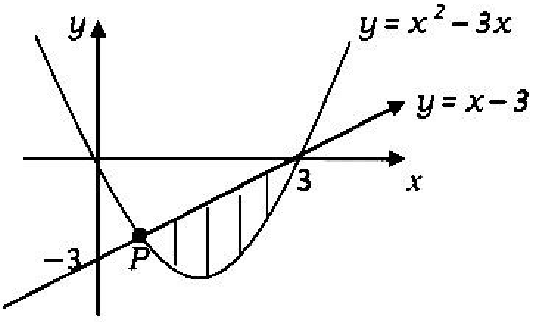
Consider the diagram given below:



$$Area = \int_{x_1}^{x_2} (f(x) - g(x))dx$$

where x_1 and x_2 are x coordinates of the points of intersection

Example The diagram given below shows the sketches of the functions $y = x - 3$ and $y = x^2 - 3x$



a) Show that the x – coordinate of point P is 1.

b) Find the area of the shaded region.

✍ **Answers**

a) Point of intersection:

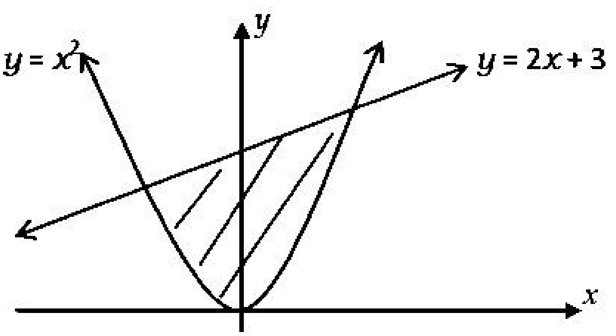
$$y_1 = y_2$$
$$x^2 - 3x = x - 3$$
$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1,\ 3$$
$$\therefore x\text{ - coordinate of point } P = 1$$

b) Area of the shaded region:

$$Area = \int_a^b (f(x) - g(x))dx$$
$$= \int_1^3 ((x-3) - (x^2 - 3x))dx$$
$$= \int_1^3 (x - 3 - x^2 + 3x)dx$$
$$= \int_1^3 (-x^2 + 4x - 3)dx$$
$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$
$$= \left[\left(-\frac{3^3}{3} + 2(3^2) - 3(3)\right) - \left(-\frac{1^3}{3} + 2(1^2) - 3(1)\right)\right]$$
$$= \frac{4}{3}\ square\ \ units$$

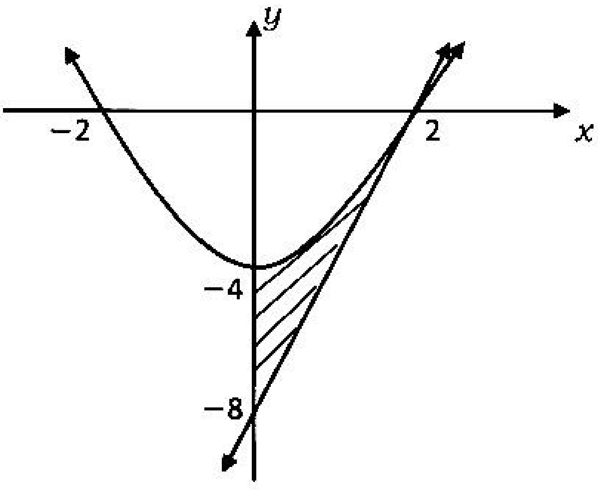
Class Activity 62

1. Two functions are shown below.

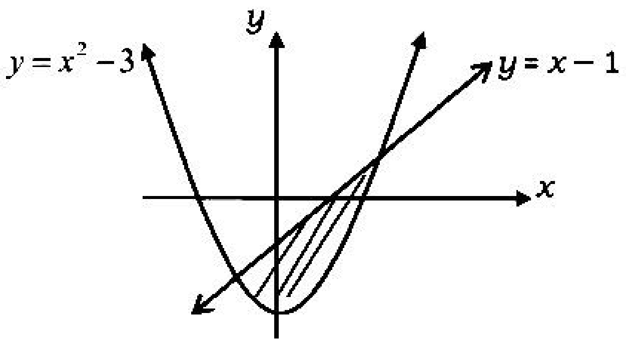


Calculate the area of the shaded region.

2. The diagram below shows the graph of the quadratic function $y = x^2 - 4$ and the straight line $y = 4x - 8$. Find the area of the shaded region.



3. The figure shown below shows the curve $y = x^2 - 3$ and the straight line $y = x - 1$



a) Calculate the x – coordinates of the point of intersection of the two graphs.

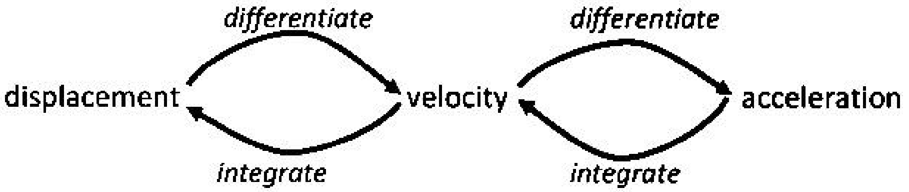
b) Calculate the area of the shaded region.

LESSON 63: **Kinematics using Integration**

Learning Outcome: Solve kinematics problems.

- **Displacement**(s) refers to straight line distance in a particular direction.
- **Velocity**(v) is the rate of change of displacement. It is a speed in a specified direction.
- **Acceleration**(a) is the rate of change of velocity.

s = displacement	measured in m
v = velocity	measured in m/s
a = acceleration	measured in m/s^2
t = time	measured in s



➢ **Common terms**

The term **initially** means when the time $t = 0$

If the velocity $v = 0$ the object is **stationary** or **at rest**. It is not moving.

Example A particle moves in a straight line so that its acceleration after t seconds is given by $a = 2t + 7$.

a) Find a formula for the velocity of the particle at time t given that initial velocity = 0.

b) Calculate the velocity at t = 3 s.

c) Find a formula for the displacement of the particle at time t given that initial displacement = 0.

d) Calculate the displacement after 9 seconds.

Answers

a) Acceleration is given we want the formula for velocity, v. So we need to integrate.

$$v = \int 2t + 7\ dt$$

$$= \frac{2t^2}{2} + 7t + c$$

$$= t^2 + 7t + c$$

Find c by noting that when $t = 0$, $v = 0 \Rightarrow c = 0$

Thus, the formula for the velocity of the particle at time t is $v = t^2 + 7t$

b) Substitute $t=3$

$$v = t^2 + 7t$$
$$= 3^2 + 7(3)$$
$$= 30\, m/s$$

c) Integrate velocity

$$s = \int t^2 + 7t\, dt$$
$$= \frac{t^3}{3} + \frac{7t^2}{2} + c$$

Find c by noting that when $t=0$, $v=0$ $\Rightarrow$ c =0

$$\therefore s = \frac{t^3}{3} + \frac{7t^2}{2}$$

d) Substitute $t=9$

$$s = \frac{9^3}{3} + \frac{7 \times 9^2}{2}$$
$$= 526\frac{1}{2} m$$

Class Activity 63

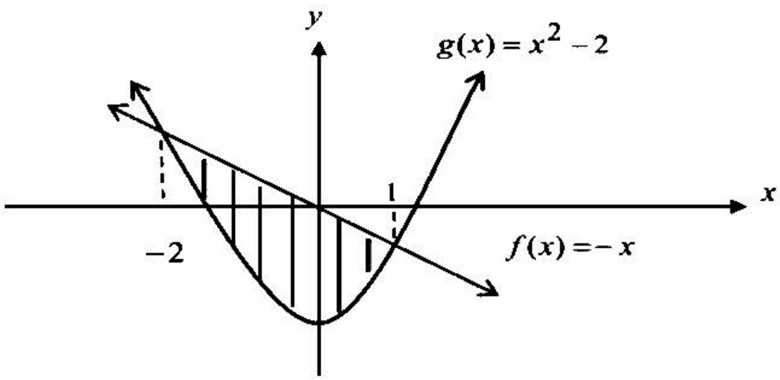
1. A body moves with a velocity of $5t - 2$ m/s, where t is the time in seconds. Find the distance the body moves in the first 5 seconds.

2. The acceleration of a body from a fixed point P is given by $a = 4t - 11$. The body is initially 2m from P and has the velocity of 11m/s after 6s.
   a) Find the formula for velocity.
   b) Find the velocity of the body after 5 seconds.
   c) When is the body at rest?
   d) Find the formula for distance travelled from point P.
   e) How far from P is the body after 10s?

3. A rock is thrown vertically upwards with an initial velocity of 40m/s from a point 5 m above the ground level. The velocity of the rock after t seconds is given by $v = 40 - 10t$.
   a) Find a formula for the height of the rock above the ground at time t.
   b) Find the maximum height reached by the rock.
   c) Calculate the distance travelled in the **third** second.

RATU NAVULA COLLEGE
YEAR 13 MATHEMATICS WORKSHEET—WEEK 8

STRAND 9 **INTEGRATION**

1.	Find $\int \frac{-9\cos 5x}{7}\, dx$ $\left(\int \cos x\, dx = \sin x \right)$
2.	A particle travels in a straight line. Its acceleration, a m/s, after t seconds is given by $a(t) = 3t^2 - 5$ Find an expression for **velocity**, v, given that when $t = 0$, $v = 10$ m/s.
3.	The shaded region in the diagram below is enclosed by the functions $f(x) = -x$ and $g(x) = x^2 - 2$  (a) Write an **expression** for the **area** of the shaded region. (b) Hence, or otherwise, determine the **area** of the shaded region.
4.	Determine $\int \frac{x^2}{\sqrt{x^3+9}}\, dx$

THE END