RATU NAVULA COLLEGE

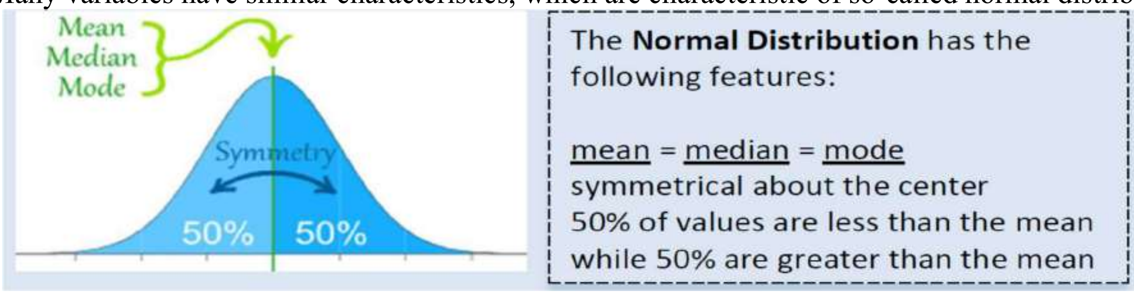
YEAR 12 Mathematics Lesson Notes – Week 9

Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

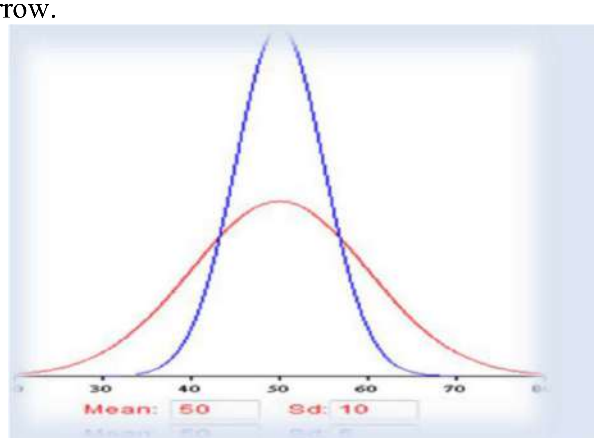
Lesson 80: Characteristics of a Normal Distribution

Learning Outcome: State characteristics of normal distribution.

- It is bell-shaped with a single peak in the center, and it is symmetrical.
- If the distribution is perfectly symmetrical with a single peak in the center, then the mean value, the mode, and the median will be all the same.
- Many variables have similar characteristics, which are characteristic of so-called normal distributions.

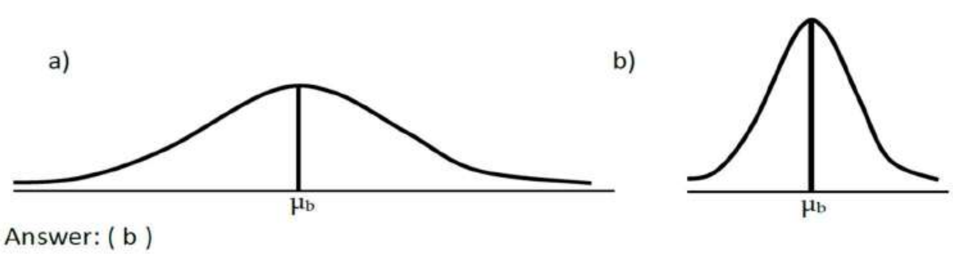


- When the standard deviation is large, the curve is short and wide; when the standard deviation is small, the curve is tall and narrow.



Example 1

Which of the following normal probability distribution has the smallest standard deviation?



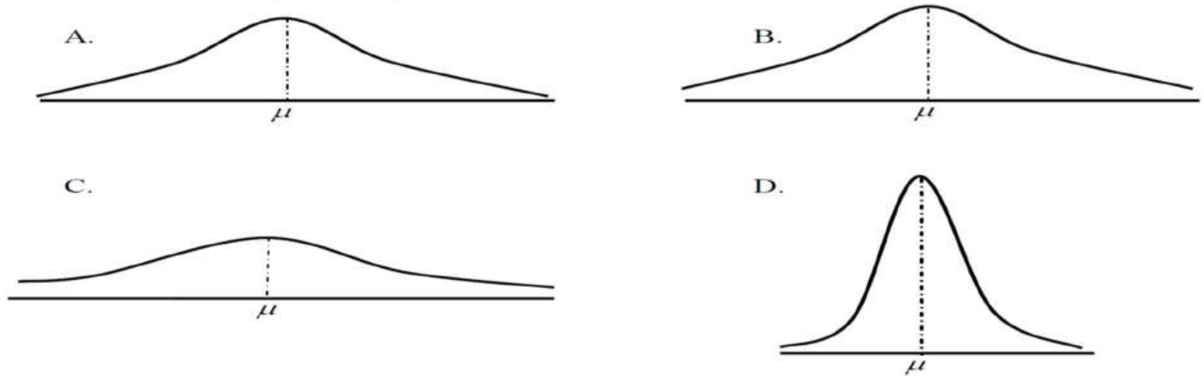
Activity

Multiple Choice

1. In a normal distribution the mean is 20. The value of the median of this distribution is

A.	0	C.	20
B.	1	D.	30

2. Which of the following normal probability distribution has the smallest standard deviation?

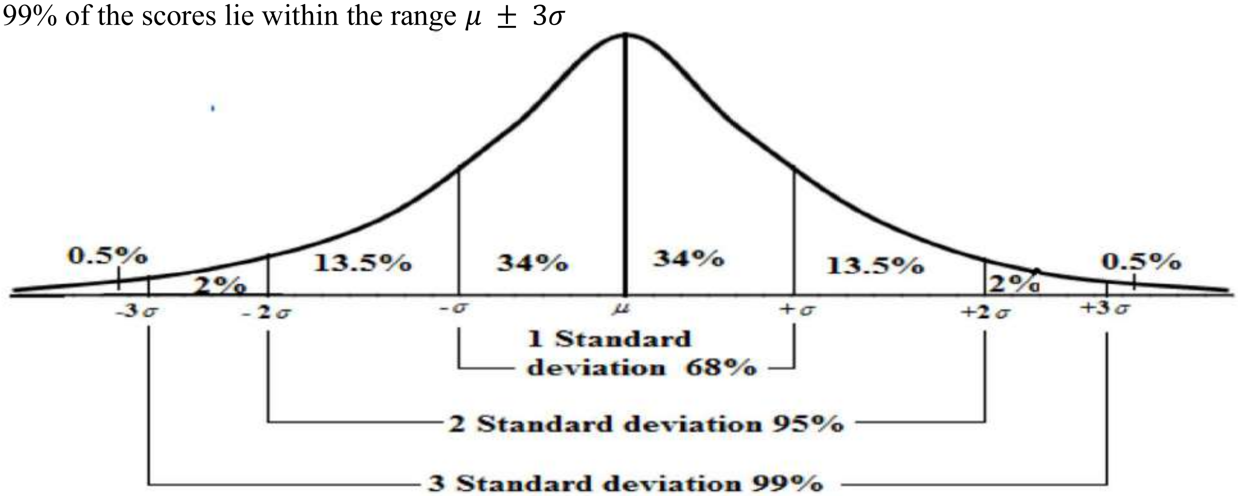


Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

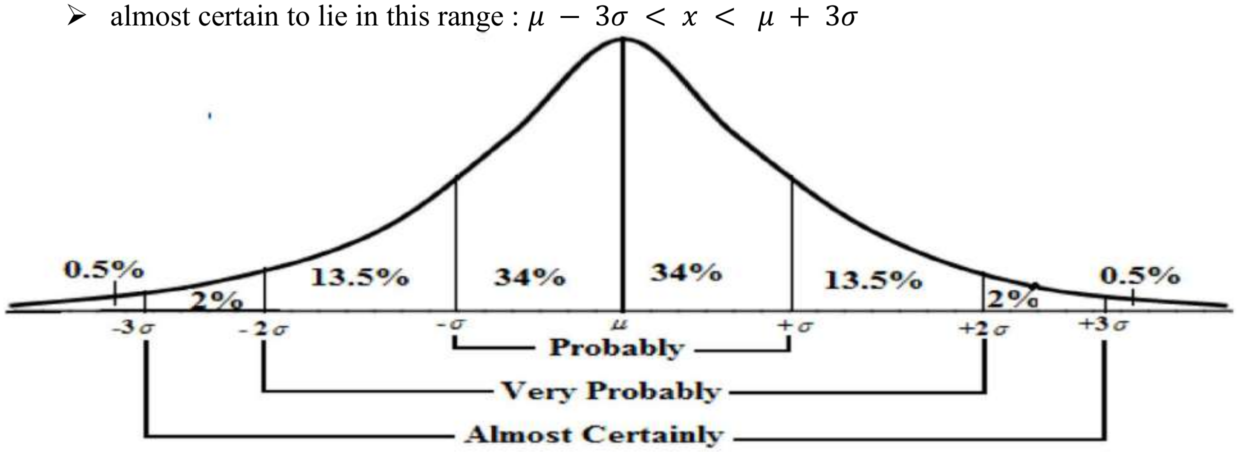
Lesson 81: Normal Distribution Terms

Learning Outcome: Define normal distribution terms.

- The normal distribution is a continuous probability distribution.
- The total area under the normal curve is equal to one
- 68% of the scores lie within the range $\mu \pm 1\sigma$
- 95% of the scores lie within the range $\mu \pm 2\sigma$
- 99% of the scores lie within the range $\mu \pm 3\sigma$



- A score is
  - probable or likely to lie in this range : $\mu - 1\sigma < x < \mu + 1\sigma$
  - very likely or very probably to lie in this range : $\mu - 2\sigma < x < \mu + 2\sigma$
  - almost certain to lie in this range : $\mu - 3\sigma < x < \mu + 3\sigma$



Example 1

Your Company packages sugar in one kg bags. When you weigh a sample of bags you get these results:

1007g, 1032g, 1002g, 983g, 1004g, 1040g, 1021g, 999g, 1009g, x g

(a) Find the mean weight of 9 bags of sugar.
(b) Another weight x g is to be included such that the mean weight is 1010g. Find the weight of the 10th bag.
(c) Work out the standard deviation.
(d) Assume that the weight of the sugar is normally distributed with a mean of 1010g standard deviation of 16g. Draw the normal distribution curve.
(e) Between what weights is the bag likely to be?
(f) Between what weights is the bag almost certain to be?

a) $\bar{x} = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \frac{1007 + 1032 + \ldots\ldots + 1009}{9} = \frac{9097}{9} = 1010\frac{7}{9}g$

b) Another weight is added so n is now 10 .

$$\bar{x} = \frac{Total}{n}$$

$$1010 = \frac{Total + x}{10}$$

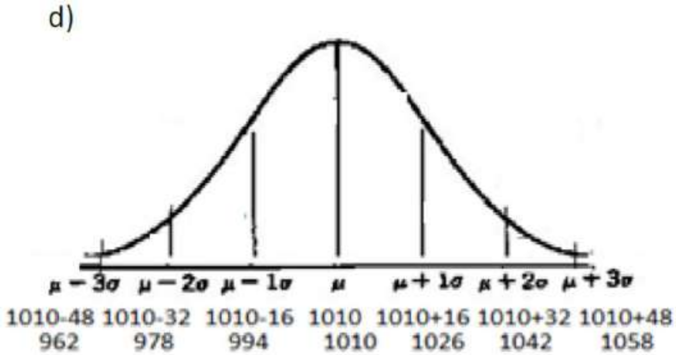
$$1010 \times 10 = 9097 + x$$
$$x = 10100 - 9097$$
$$\therefore x = 1003g$$

c) Put calculator mode on SD. Press each number with M+ sign.

Shift | 2 | 2 | =

$\sigma = 15.86 \approx 16g$

d)



e) **likely** to be:
$\mu - 1\sigma < x < \mu + 1\sigma$
$994g < x < 1026g$

f) **almost certainly**:
$\mu - 3\sigma < x < \mu + 3\sigma$
$962g < x < 1058g$

Activity

Multiple Choice

1. A normal distribution has a mean of 10 and standard deviation of 2. A score selected at random will almost certainly lie between

   A. 4 and 16   C. 8 and 12
   B. 6 and 14   D. 8 and 16

2. A set of quiz scores is normally distributed with mean = 50 and standard deviation = 5. A score selected at ra ndom will almost certainly lie between

   A. 35 and 50   C. 40 and 60
   B. 35 and 65   D. 45 and 55

3. A set of quiz scores is normally distributed with mean = 10 and standard deviation = 2. A score selected at random is likely to lie between

   A. 8 and 12   C. 4 and 16
   B. 6 and 14   D. 10 and 16

Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

Lesson 82: Standard Normal Distribution

Learning Outcome: Process of Standardizing (converting to z-score)

Each set of data that is normally distributed has a different mean and standard deviation. It is therefore impossible to give probabilities for every situation. Instead, we convert to standard normal distribution. It is a normal distribution with mean of 0 standard deviation of 1.

To convert a value to a Standard Score:

- first subtract the mean,
- then divide by the Standard Deviation

$$\boxed{z = \frac{x-\mu}{\sigma}}$$

For negative z – score

- use the positive equivalent as the normal curve is a symmetrical graph.

For clarity purpose, draw and shade the required region.

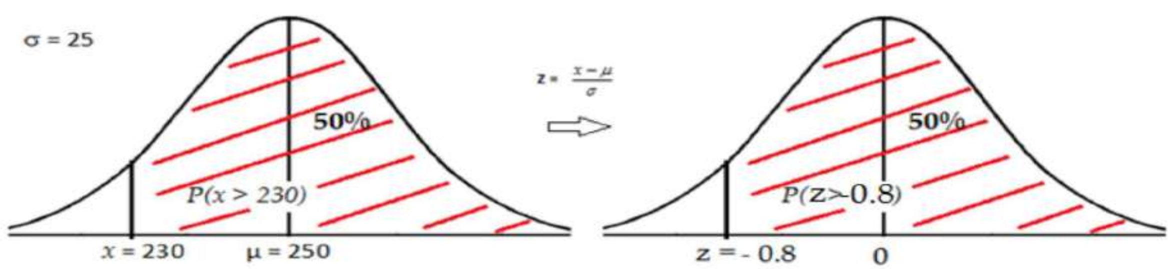
Example 1

For the following scores, convert it to the z – score and draw the required region.

(a) mean $\mu = 250$
Standard Deviation $\sigma = 25$
$x = 230$
$P(x > 230)$

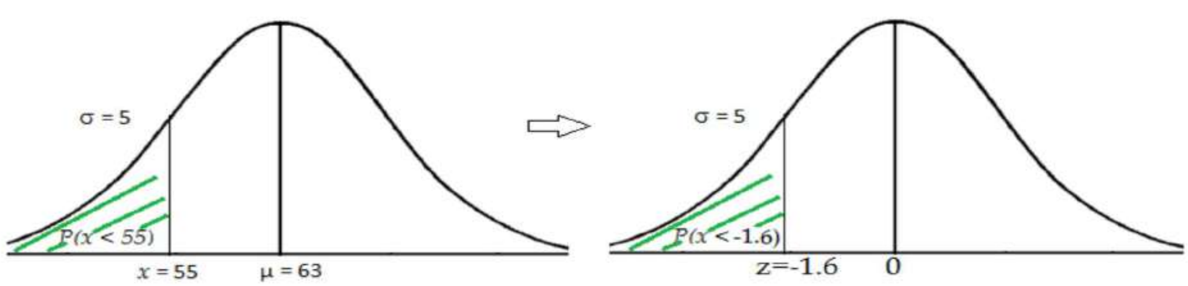
a)

$$Z = \frac{x-\mu}{\sigma} = \frac{230-250}{25} = -0.8$$



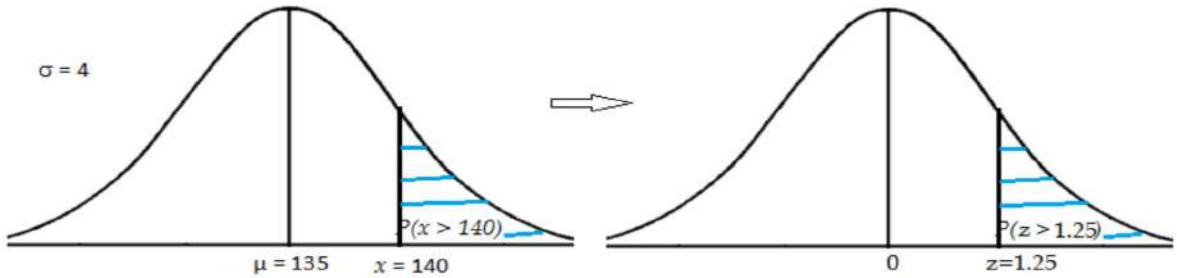
(b) mean $\mu = 63$
Standard Deviation $\sigma = 5$
$x = 55$
$P(x < 55)$

b) $Z = \frac{x-\mu}{\sigma} = \frac{55-63}{5} = -1.6$



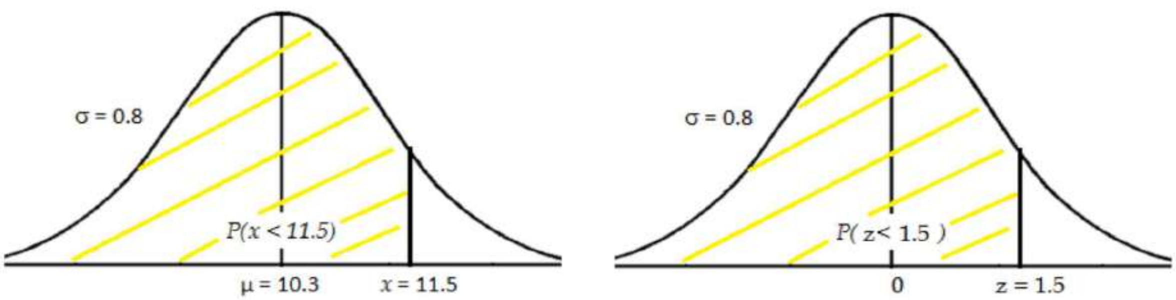
(c) mean $\mu = 250$
Standard Deviation $\sigma = 25$
$x = 230$
$P\,(x > 230)$

c) $Z = \dfrac{x-\mu}{\sigma} = \dfrac{140-135}{4} = 1.25$



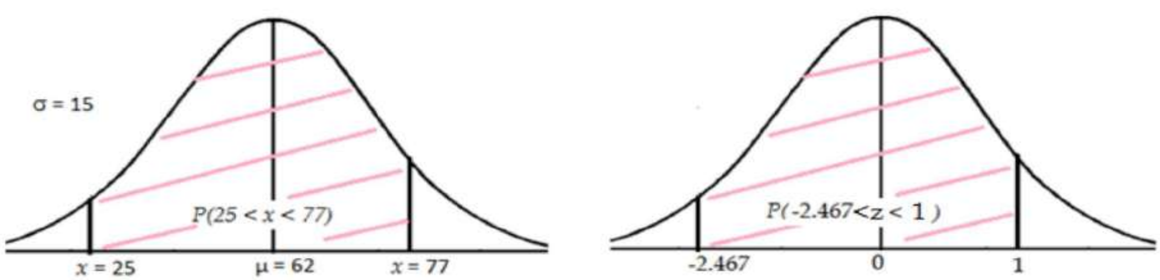
(d) mean $\mu = 10.3$
Standard Deviation $\sigma = 0.8$
$x = 11.5$
$P\,(x < 11.5)$

$$Z = \frac{x-\mu}{\sigma} = \frac{11.5-10.3}{0.5} = 1.5$$



(e) mean $\mu = 62$
Standard Deviation $\sigma = 15$
$x = 25 \qquad x = 77$
$P\,(25 < x < 77)$

b) $Z = \dfrac{x-\mu}{\sigma} = \dfrac{25-62}{15} = -2.467$, $Z = \dfrac{x-\mu}{\sigma} = \dfrac{77-62}{15} = 1$



Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

Lesson 83: Finding Probabilities

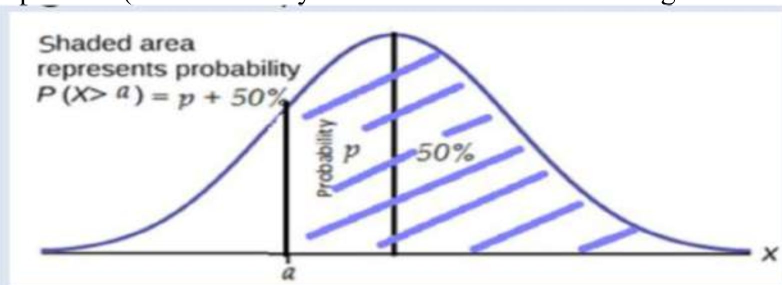
Learning Outcome: Determine probabilities from normal distribution table.

The probability that a normal random variable *X* equals any particular value is 0.

- Less than a (option "Up to a ")
- Greater than a (option " a onwards")

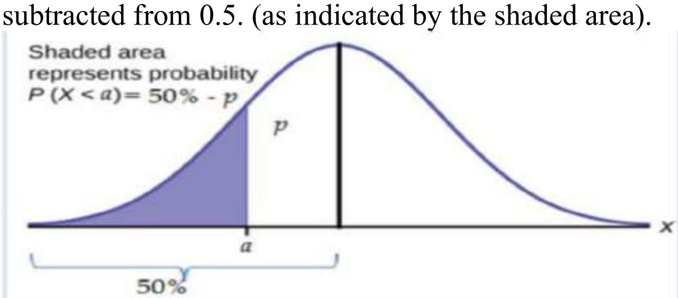
Consider the probabilities of different regions:

The probability that ***x* is greater than a** on the left side is equal to the area under the normal curve bounded by a plus 0.5 (as indicated by the non-shaded area in the figure below).



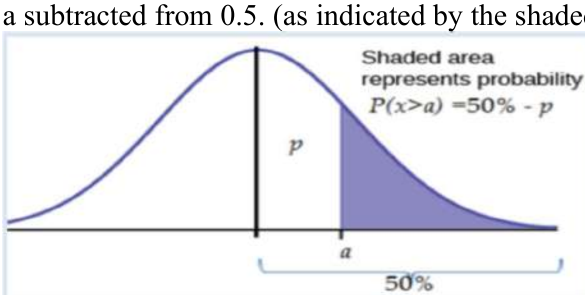
P(x > a) = 0.5 + p

The probability that ***x* is less than a** on the left side is equal to the area under the normal curve bounded by a subtracted from 0.5. (as indicated by the shaded area).



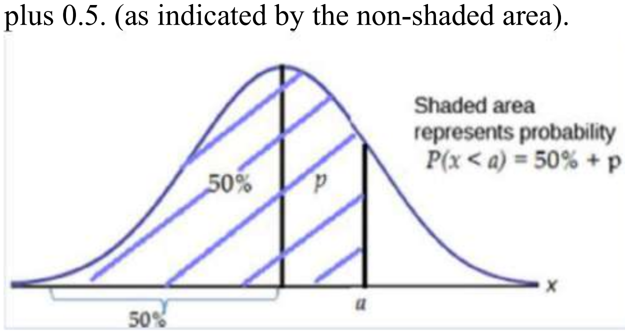
P(x < a) = 0.5 − p

The probability that ***x* is greater than a** on the right side is equal to the area under the normal curve bounded by a subtracted from 0.5. (as indicated by the shaded area).



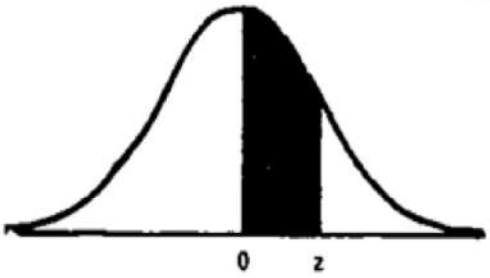
P(x < a) = 0.5 − p

The probability that ***x* is less than a** on the left side is equal to the area under the normal curve bounded by a plus 0.5. (as indicated by the non-shaded area).



P(x > a) = 0.5 + p

AREAS UNDER NORMAL PROBABILITY CURVE



The tabulated value is the probability that the standardized normal variate Z (with $\mu=0$, $\sigma=1$) lies between 0 and z.
e.g. $P(0 < Z < 1.43) = 42.36\%$

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359	4	8	12	16	20	24	28	32	36
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754	4	8	12	16	20	24	28	32	36
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141	4	8	12	15	19	22	27	31	35
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549	3	6	10	13	16	19	23	26	29
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621	2	5	7	9	12	14	16	18	21
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830	2	4	6	8	10	12	14	16	19
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015	2	4	5	7	9	11	13	15	16
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177	2	3	5	6	8	10	11	13	14
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319	1	3	4	6	7	8	10	11	13
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441	1	2	4	5	6	7	8	10	11
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545	1	2	3	4	5	6	7	8	9
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633	1	2	3	3	4	5	6	7	8
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706	1	1	2	3	4	4	5	6	6
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767	1	1	2	2	3	4	4	5	5
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817	0	1	1	2	2	3	3	4	4
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857	0	1	1	2	2	2	3	3	4
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890	0	1	1	1	2	2	2	3	3
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916	0	0	1	1	1	2	2	2	2
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936	0	0	1	1	1	1	1	2	2
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952	0	0	0	1	1	1	1	1	1
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964	0	0	0	0	1	1	1	1	1
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974	0	0	0	0	0	1	1	1	1
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981	0	0	0	0	0	0	0	0	1
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986	0	0	0	0	0	0	0	0	1
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990	0	0	0	0	0	0	0	0	0
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993	0	0	0	0	0	0	0	0	0
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995	0	0	0	0	0	0	0	0	0
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997	0	0	0	0	0	0	0	0	0
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	0	0	0	0	0	0	0	0	0
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	0	0	0	0	0	0	0	0	0
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	0	0	0	0	0	0	0	0	0

Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

Lesson 84: Finding Probabilities

Learning Outcome: Determine probabilities from normal distribution table.

To find the probability of standard scores, z value is not enough. You need to know how to use the table.

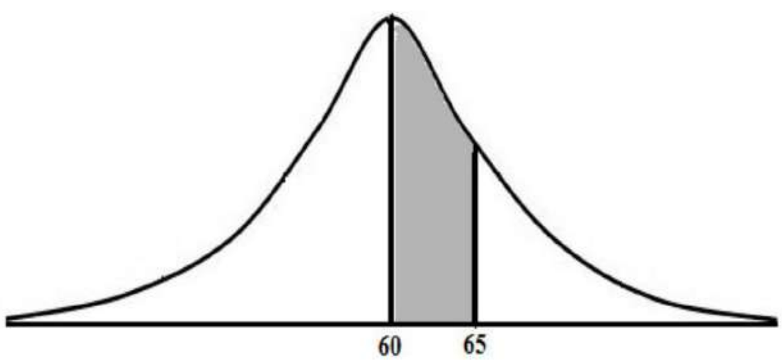
- Find the z – score with reference to question.
- Draw and shade the required region. Done previously
- Use table to find the probability in relation to z –score
- Add 0.5 or Subtract from 0.5 or subtract from 1 etc. depending on shaded area.

Example 1

5000 students sat for a Mathematics test. Their mean mark is 60 with a standard deviation of 10.

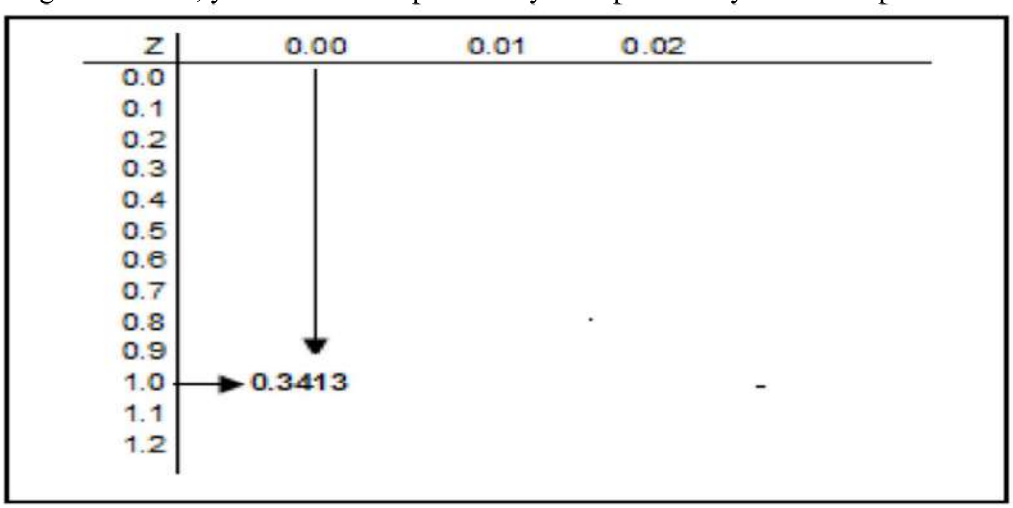
(a) Find the Z score for the mark between the mean and 65
(b) Find the probability that the mark will be between the mean and 65
(c) How many students are expected to have their marks between the mean and 65?

(a)



$Z = \frac{x-\mu}{\sigma}$ $\quad Z = \frac{65-60}{5}$ $\quad Z = \frac{5}{5}$ $\quad \underline{\underline{Z = 1}}$

(b) Using the Z score, you can find the probability. The probability that correspond to Z = 1 is 0.3413.



Z	0.00	0.01	0.02
0.0			
0.1			
0.2			
0.3			
0.4			
0.5			
0.6			
0.7			
0.8			
0.9			
1.0	0.3413		
1.1			
1.2			

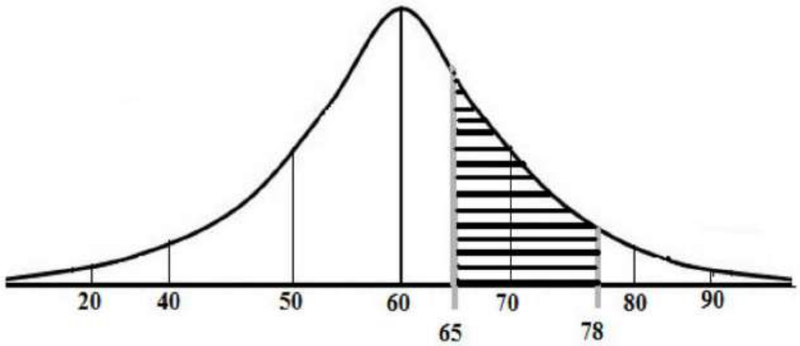
This means that the probability that students mark will be between the mean and 65 is 0.3413

(c)

$$\boxed{Given \rightarrow \boxed{n = 5000} \; \boxed{P = 0\cdot3413}}$$

$$\boxed{Expected\ number = np \Rightarrow 5000 \times 0\cdot3413} \rightarrow \boxed{\boxed{\approx \underline{\underline{1707\ \ students}}}}$$

Example 2
40 students sat for Physics test and their mean mark was 60 and the standard deviation was 10.
Find the probability that the students mark will be between 65 and 78.



You won't be able to find the probability between 65 and 78 directly. You will need to use the Z-score.

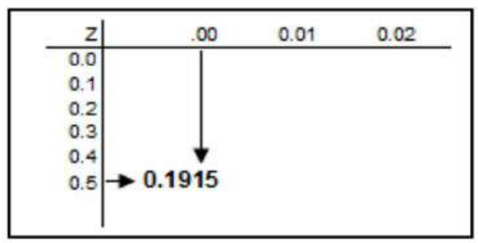
However, Z-score gives the probability between the mean and the x-value.

- Step one. Find the probability between 60 and 65.
- Step two. Find the probability between 60 and 78
- Step three. Step two answer – Step one answer.

STEP ONE:

$Z = \dfrac{x-\mu}{\sigma}$ $\quad Z = \dfrac{65-60}{10}$ $\quad Z = \dfrac{5}{10}$ $\quad Z = 0\cdot 5$

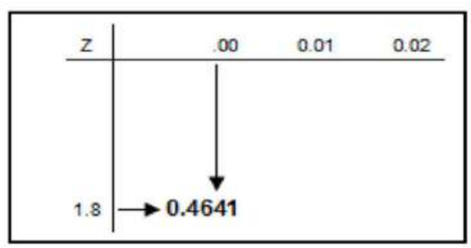
The probability between 60 and 65 is $\underline{\underline{0\cdot 1915}}$



STEP TWO:

$Z = \dfrac{x-\mu}{\sigma}$ $\quad Z = \dfrac{78-60}{10}$ $\quad Z = \dfrac{18}{10}$ $\quad Z = 1\cdot 8$

The probability between 60 and 78 is $\underline{0\cdot 4641}$

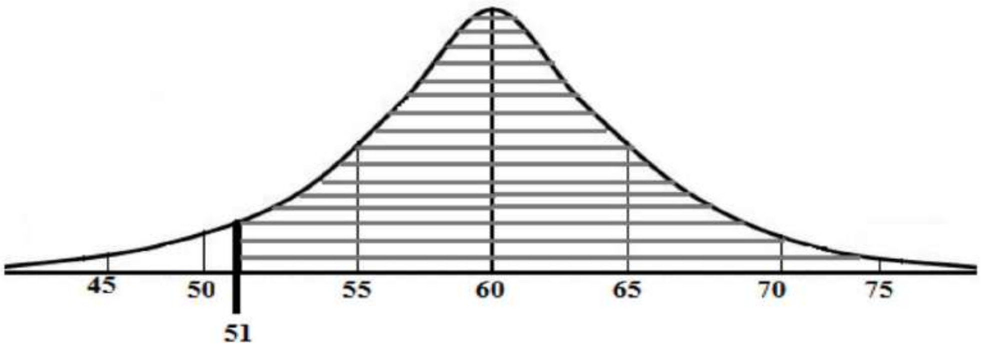


STEP THREE:

$0\cdot 4641 - 0\cdot 1915 = 0\cdot 2726$

The probability that a students mark will be between 65 and 78 is $\underline{\underline{0\cdot 2726}}$

Example 3
The Fiji Seventh Form Examination of a particular school had a mean of the Mathematics mark equal to 60 with a standard deviation equal to 5.
Find the probability that a student's mark will be more than 51.



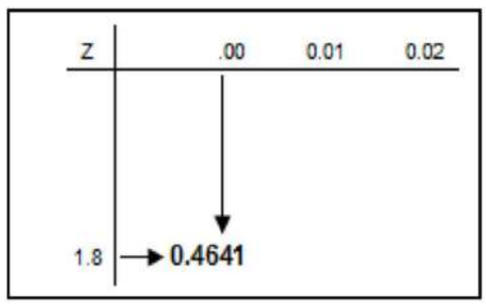
- Step One: Find probability between the mean 60 and the x-value 51
- Step Two: Add 0.5 to the answer of step one.

STEP ONE:

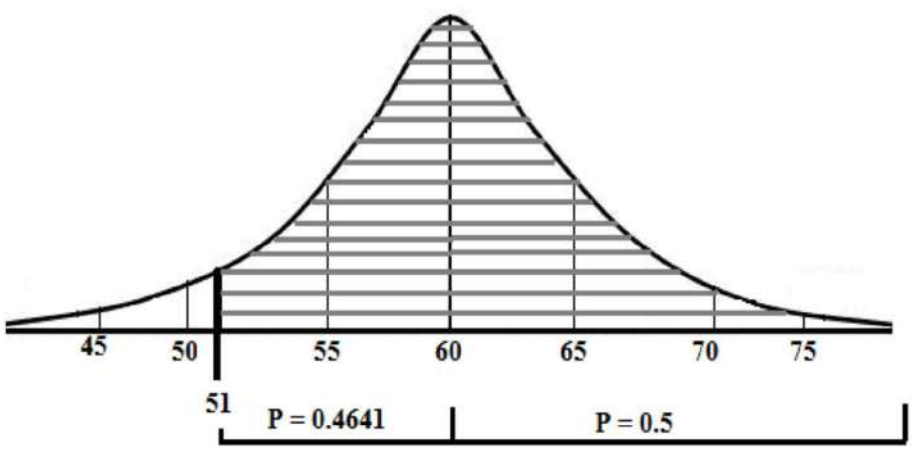
Z-score helps us to find the probability between the mean and the x-value. Here the mean is 60 and the x-value is 51.

$$\boxed{Z=\frac{\mu-x}{\sigma}} \rightarrow \boxed{Z=\frac{60-51}{5}} \rightarrow \boxed{Z=\frac{9}{5}} \rightarrow \boxed{Z=1\cdot 8}$$

The probability between the mean 60 and the x-value 51 is $\underline{0\cdot 4641}$



STEP TWO:



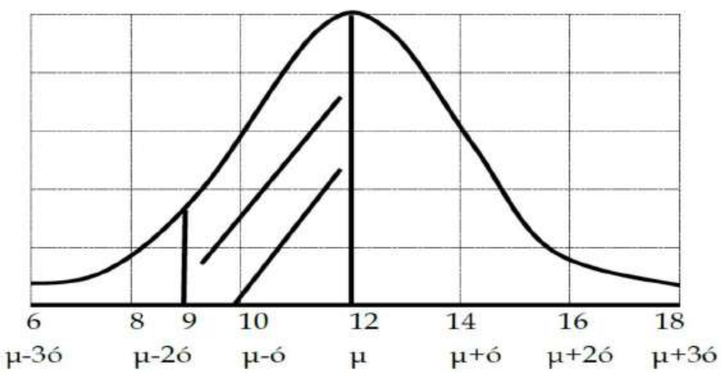
The graph on the left of the mean has a total probability equal to $0\cdot 5$.
$0\cdot 5+0\cdot 4641=\underline{\underline{0\cdot 9641}}$

Example 4

Cargo containers are unloaded from a ship by a small crane and stored on the wharf. The weight of the containers is normally distributed with a mean of 12 tonnes and standard deviation of 2 tonnes. The maximum weight that the small crane can lift is 16 tonnes. If a container is heavier than 16 tonnes, a special heavy lift crane must be brought over to lift the container.

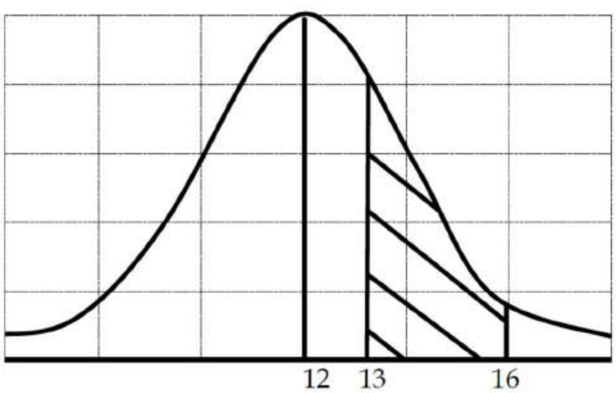
(i) A container is lifted at random. What is the probability that the weight of the container is between 9 tonnes and 12 tonnes?
(ii) What is the probability that a container lifted by the small crane weighs more than 13 tonnes?
(iii) The ship has 250 containers on it. Approximately how many containers will need to be lifted by the special heavy lift crane?

(i)



$$Z=\frac{x-\mu}{\sigma}=\frac{9-12}{2}=-1.5$$
$$P(9<x<12)=0.4332$$

(ii)

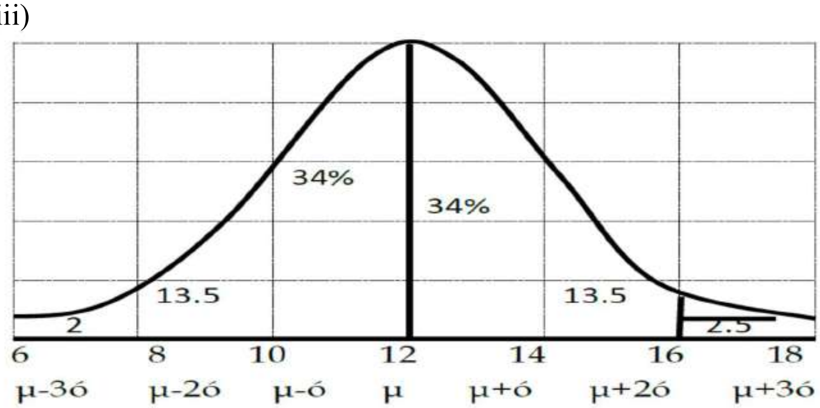


$$Z=\frac{x-\mu}{\sigma}=\frac{13-12}{2}=0.5$$
$$P(12>x>13)=0.1915$$

$$Z=\frac{x-\mu}{\sigma}=\frac{16-12}{2}=2$$
$$P(12>x>16)=0.4772$$

$$Z=\frac{x-\mu}{\sigma}=\frac{13-16}{2}=1.5$$
$$P(13>x>16)=0.4772-0.1915=0.2857$$

(iii)



$$E=n.p=250\times 2.5\%$$
$$=250\times 0.025=6.25$$
$$\approx 6 containers$$

Example 5

The lifetime of a newly designed light bulb is a normally distributed variable with a mean of 18 hrs and standard deviation of 2 hrs.

(i) What is the probability that a light bulb tested at random will last between 18 hrs to 21 hrs?

(ii) What percentage of the light bulbs is expected to last longer than 23 hrs?

(iii) Estimate how many light bulbs should be purchased to have an expected number of 100 that will last at least for 16 hrs.

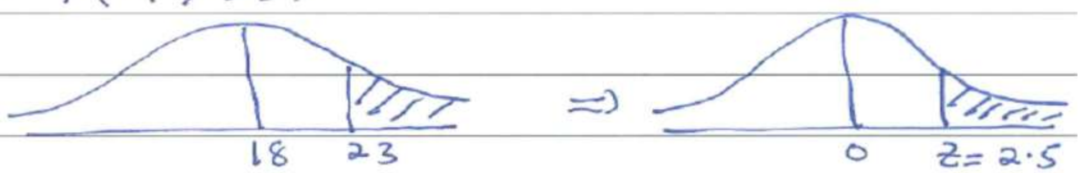
(i) $\bar{x} = 18\text{hrs} \qquad \sigma = 2\text{hrs}$

$P(18 < x < 21)$

$z = \frac{x - \bar{x}}{\sigma} = \frac{21 - 18}{2} = 1.5$

$P(18 < x < 21) = \underline{0.4332}$

(ii) $P(x > 23)$



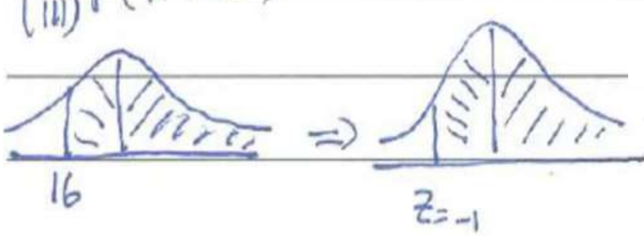
$z = \frac{23 - 18}{2} = 2.5$

$P(x > 23) = 0.5 - P(z = 2.5)$

$= 0.5 - 0.4938 \quad \text{or} \quad 0.0062$

$= \underline{\underline{0.62\%}}$

(iii) $P(x > 16)$



$0.3413 + 0.5$

$= 0.8413$

$E = np$

$100 = n(0.8413)$

$n = \frac{100}{0.8413} = 118.9$

$\underline{\underline{n \simeq 119}}$ light bulbs

Activity

1. The crushing loads of identically sized samples of concrete are normally distributed with a mean of 7.4 kN and a standard deviation of 0.8 kN.

   (a) What is the probability that a randomly selected concrete sample will have a crushing load of less than 5.2 kN?

   (b) Concrete with a crushing load of less than 5.2 kN is rated as unsatisfactory. From a random sample of 1000 concrete, how many would be expected to be unsatisfactory?

2. The weights of Year 12 students are normally distributed with a mean of 60 kg and a standard deviation of 4 kg.

   (a) What is the probability that a randomly selected Year 12 student weighs more than 55 kg? Give your answer to 4 decimal places.

   (b) From a sample of 2500 Year 12 students, how many may be expected to weigh more than 55kg?

3. The heights of Year 12 students are normally distributed with a mean of 160 cm and a standard deviation of 4 cm.

   (a) What is the probability that a randomly selected Year 12 student has the height between 149 cm and 171 cm? Give your answer correct to 3 decimal places.

   (b) From a sample of 1000 Year 12 students, how many may be expected to have the height between 149 cm and 171 cm?

4. The lengths of a sample of fish are normally distributed with mean 30 cm and standard deviation 5 cm.

   (a) What is the probability that a randomly selected fish from this sample is less than 32.2 cm?

   (b) From a sample of 100 fish, how many may be expected to be less than 32.2 cm?

5. A soft drink company finds that a machine fill cans with a mean of 500 ml and a standard deviation of 5 ml. The volume of all the cans produced is normally distributed.

   (a) What is the probability that a can selected at random contains more than 498.1 ml? Give your answer to 3 decimal places.

   (b) From a sample of 2000 cans, how many may be expected to contain more than 498.1 ml?

RATU NAVULA COLLEGE

YEAR 12 MATHEMATICS - WORKSHEET 9

Strand 8: Probability **Sub Strand 8.2: Normal Distribution**

1. A study is made on how long certain batteries last. The results follow a normal distribution with a mean of 500 hours and standard deviation of 40 hours.

   (a) A battery is selected at random. What is the probability it will last between 400 and 600 hours? Give your answer to 4 decimal places.

   (b) A battery which lasts less than 376 hours is called a failure. What is the probability of getting a failure? Give your answer to 3 decimal places.

   (c) Each failure costs $50 to be replaced. From 7000 batteries, what would be the expected cost of replacing the failures?

2. The lifespan of a brand new flat screen TV is normally distributed with a mean of 8 years and a standard deviation of 2 years.

   (a) What is the probability that a brand new flat screen TV selected at random will have a lifespan between 9.5 and 12.5 years?

   The company that manufactures the brand new flat screen TV gives warranty period of 2 years for the TV screen. If the TV screen fails within this period, it will be replaced by the company.

   (b) If the company manufactures 2 250 TV screens, approximately how many of this brand new flat screen TV will be replaced?

3. A vegetable farmer in the Western Division supplies a large number of capsicums also known as 'bell pepper' to a national supermarket chain. The weight of the capsicums is normally distributed, with mean 185 grams and standard deviation 30 grams.

   (a) What is the probability that a capsicum chosen at random weighs more than 170 grams?

   (b) What is the probability that a capsicum chosen at random weighs between 215 grams and 230 grams?

   (c) Estimate the number of capsicums in a consignment of 600 that could be expected to weigh over 250 grams.

4. The weights of 100 senior citizens are normally distributed with a mean weight of 65 kg and a standard deviation of 8 kg.

   (a) What is the probability that a senior citizen selected at random has aweight between 65 and 75 kg ? Give your answer correct to 4 decimal places.

   (b) Senior citizens that have weights lower than 53 kg are put on a special diet. How many senior citizens are expected to be put on a special diet ?

   (c) Weights which are beyond 2 standard deviations of the mean are classified as undesirable. What percentage of the senior citizens are expected to have undesirable weights ? Give your answer to the nearest whole number.