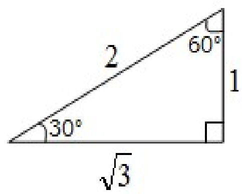
RATU NAVULA COLLEGE

YEAR 12 Mathematics Lesson Notes – Week 4

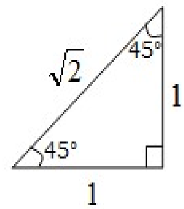
Strand 5: Trigonometry **Sub Strand 5.1: Non-Right Angled Triangles**

Lesson 55: Special Triangles

Learning Outcome: Find the Exact Value Of Sine/Cosine/Tangent and also find the area of the triangles.



30°- 60° - 90° Triangle



Isosceles Right-Angled Triangle

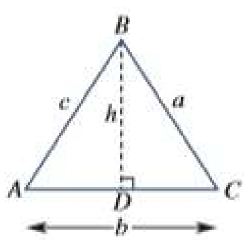
Some Common Trigonometric Ratios

Angle (θ)	0°	30°	45°	60°	90°
$Sin\ \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$Cos\ \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$Tan\ \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Area of Triangle

The area of a triangle can be calculated in a triangle without a right angle. It requires that two sides and the included angle are known. The area of a triangle is half the product of two sides multiplied by the sine of the angle between the two sides (included angle).

This result is derived by constructing a perpendicular line of length h from B to D.



Area of a triangle is calculated using the formula:

$$A = \tfrac{1}{2}bh$$

In ΔBCD, $\sin C = \dfrac{h}{a}$

$$h = a \sin C$$

Substituting $a \sin C$ for h into $A = \frac{1}{2}bh$

$$A = \tfrac{1}{2}ba \sin C$$
$$= \tfrac{1}{2}ab \sin C$$

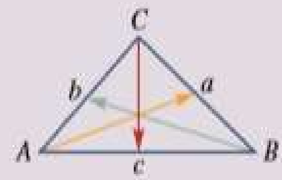
Similarly, by constructing perpendiculars from C and A, we can obtain the other two results below.

AREA OF A TRIANGLE

$A = \frac{1}{2}bc \sin A$

$A = \frac{1}{2}ac \sin B$

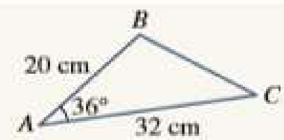
$A = \frac{1}{2}ab \sin C$



Area of a triangle is half the product of two sides multiplied by the sine of the angle between the two sides (included angle).

Example 1

Find the area of the triangle to the nearest square centimetre.

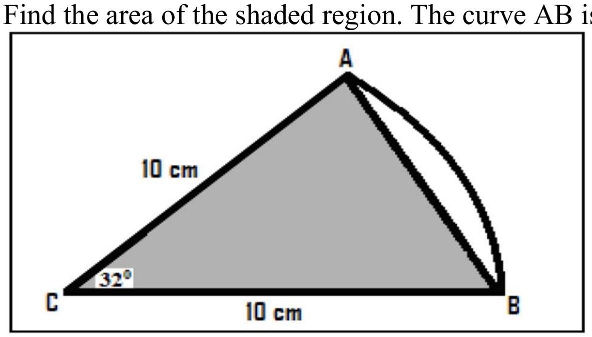


SOLUTION:

1. Write the formula. $A = \frac{1}{2}bc \sin A$
2. We are given two sides b, c and the angle A between these sides (included angle).
3. Substitute values for b, c and A. $= \frac{1}{2} \times 32 \times 20 \times \sin 36^\circ$
4. Evaluate. $= 188.0912807$
5. Write, correct to the nearest square centimetre. $= 188 \text{ cm}^2$

Example 2

Find the area of the shaded region. The curve AB is part of a circle with r = 10 cm.



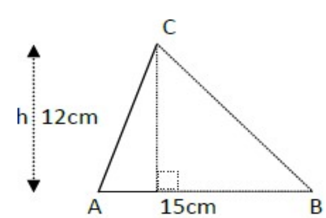
$$A = \frac{1}{2} a\, b\, Sin\, C$$

$$A = \frac{1}{2}(10cm)(10cm)Sin\, 32^{o}$$

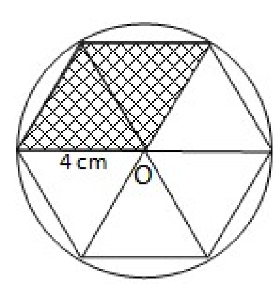
$$A = 26 \cdot 50\, cm^2$$

Activity

1. Calculate the area of triangle ABC shown.



2. The diagram below shows a regular hexagon inscribed in a circle of radius 4 cm at centre O. (Diagram not to scale).
   Calculate:
   (i) the angle of each sector formed.
   (ii) the area of one of the triangles.
   (iii) the area of shaded region.



Strand 5: Trigonometry **Sub Strand 5.1: Non-Right Angled Triangles**

Lesson 56: Angles

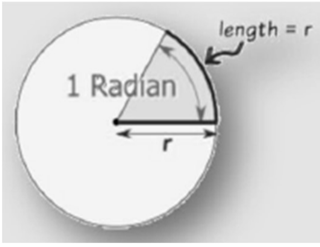
Learning outcome: Change radians to degree and degree to radians.

Angles

- It can be measured in two ways:
  - Degrees
  - Radians

Radians

- One radian is the angle subtended by an arc equal in length to the radius.



There are 2π radians in one revolution.

2π radians	=	360˚
π radians	=	180˚
$\frac{\pi}{2}$radians	=	90˚
$\frac{\pi}{4}$radians	=	45˚

Converting Degrees to Radians

$$Radians = \frac{\theta^{\circ}}{180^{\circ}} \times \pi$$

Example

Convert the following degrees to radians.

a. $135^{\circ} = (135 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = \frac{3\pi}{4}$ radians — Multiply by $\pi/180$.

b. $540^{\circ} = (540 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = 3\pi$ radians — Multiply by $\pi/180$.

c. $-270^{\circ} = (-270 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = -\frac{3\pi}{2}$ radians — Multiply by $\pi/180$.

Degree

$$360^{\circ} = 2\pi \text{ rad} \quad \text{and} \quad 180^{\circ} = \pi \text{ rad.}$$

From the latter equation, you obtain

$$1^{\circ} = \frac{\pi}{180} \text{ rad} \quad \text{and} \quad 1 \text{ rad} = \left(\frac{180^{\circ}}{\pi}\right)$$

- **<u>Converting Radians to Degrees</u>**

$$Degrees = \text{Radians} \times \frac{180^\circ}{\pi}$$

<u>Example</u>

Convert the following radians to degree.

a. $-\frac{\pi}{2}\text{ rad} = \left(-\frac{\pi}{2}\text{ rad}\right)\left(\frac{180\text{ deg}}{\pi\text{ rad}}\right) = -90^\circ$ Multiply by $180/\pi$.

b. $\frac{9\pi}{2}\text{ rad} = \left(\frac{9\pi}{2}\text{ rad}\right)\left(\frac{180\text{ deg}}{\pi\text{ rad}}\right) = 810^\circ$ Multiply by $180/\pi$.

c. $2\text{ rad} = (2\text{ rad})\left(\frac{180\text{ deg}}{\pi\text{ rad}}\right) = \frac{360^\circ}{\pi} \approx 114.59^\circ$ Multiply by $180/\pi$.

<u>Activity</u>

1. Convert the following radians to degree:
   (a) 2.4π

   (b) $\frac{7}{3\pi}$

   (c) 3π

2. Convert the following to degrees to radian:
   (a) $45°$

   (b) $135°$

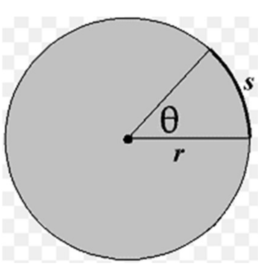
   (c) $125°$

<u>Strand 5: Trigonometry</u> **<u>Sub Strand 5.1: Non-Right Angled Triangles</u>**

<u>Lesson 57: Circular Measure</u>

Learning outcome: Find the arc length and area of sector.

1. **<u>Arc Length</u>**

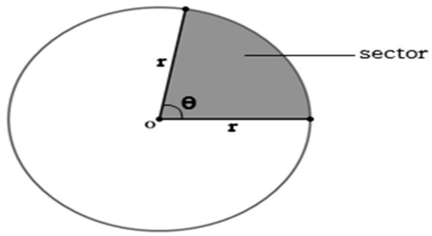


$$S = r\theta$$

where S = arc length
r = radius
θ = angle in radians

2. **Area of Sector**

Sector is the area bounded or enclosed by an arc and two radii.

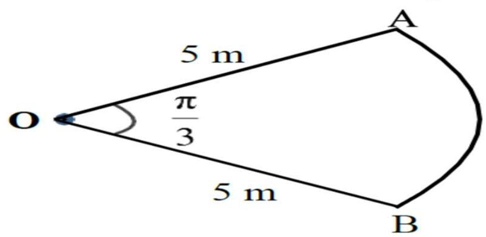


$$A_{sector} = \frac{1}{2} r^2 \theta$$

where r = radius
θ = angle in radians

Example 1

A garden in the form of a sector with centre O, radius 5 m and angle $\frac{\pi}{3}$ radians is shown below.



The garden is to be fenced on all sides. How much fencing wire will be needed?

Adding arc and two radii length:

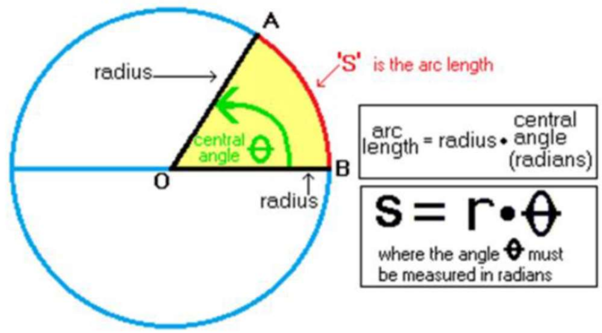
$$S = r\theta$$

$$S = 5 \times \frac{\pi}{3}$$

$$S = \frac{5\pi}{3}$$

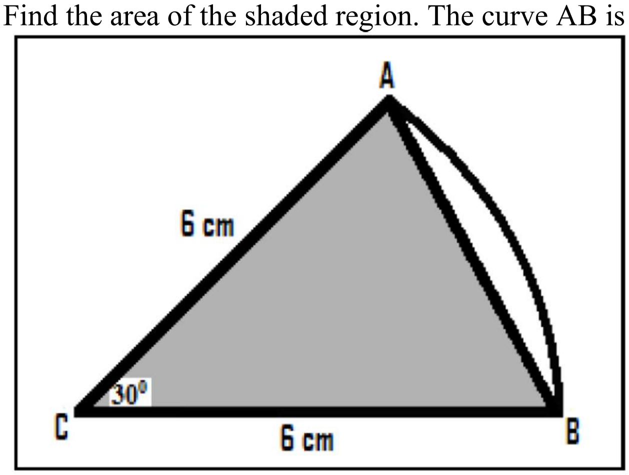
$$\therefore \quad P = \frac{5\pi}{3} + 10 \text{ or } 15.24 \text{ m}$$

Arc length formula:



Example 2

Find the area of the shaded region. The curve AB is part of a circle with r = 6 cm.



$$Area = \frac{1}{2} ab \sin C$$

$$Area = \frac{1}{2}(6cm)(6cm)\sin 30^0$$

$$\boxed{Area = 9cm^2}$$

Example 3

A sprinkler on a golf course fairway is set to spray water over a distance of 70 feet and rotates through an angle of 120°. Find the area of the fairway watered by the sprinkler.

First convert 120° to radian measure as follows.

$$\theta = 120°$$
$$= (120 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) \qquad \text{Multiply by } \pi/180.$$
$$= \frac{2\pi}{3} \text{ radians}$$

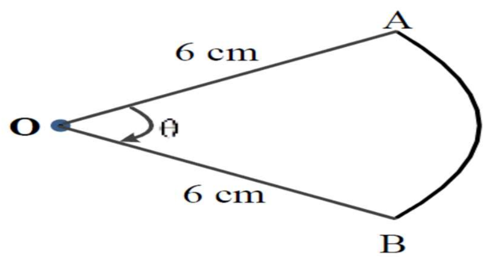
Then, using $\theta = 2\pi/3$ and $r = 70$, the area is

$$A = \frac{1}{2}r^2\theta \qquad \text{Formula for the area of a sector of a circle}$$
$$= \frac{1}{2}(70)^2\left(\frac{2\pi}{3}\right) \qquad \text{Substitute for } r \text{ and } \theta.$$
$$= \frac{4900\pi}{3} \qquad \text{Simplify.}$$
$$\approx 5131 \text{ square feet.} \qquad \text{Simplify.}$$

FIGURE 4.19

Example 4

The diagram below shows a sector of a circle with the radius of 6 cm. Point O is the centre of the circle.



(i) If the area of the sector OAB is 3π cm^2 , find θ.

(ii) Calculate the length of the minor arc AB.

(i)

$$A = \frac{1}{2}r^2\theta$$
$$3\pi = \frac{1}{2}\times 6^2 \times \theta$$
$$\theta = \frac{3\pi \times 2}{36}$$

Substitute the values into the formula:
$Area \quad of \quad Sector = \frac{1}{2}r^2\theta$

Area = 3π , radius = 6cm

Solve to find the value of angle Θ.

$$\theta = \frac{\pi}{6} rad \text{ or } 30°$$

(ii)

$$S = r\theta$$
$$A = 6 \times \frac{\pi}{6}$$

Use the formula
$Length \quad of \quad Arc = S = r\theta$

Radius = 6cm, Θ = $\frac{\pi}{6} rad$

S= π or 3.14 cm

Example 5

In Fig 11.4, a circle of radius 7.5 cm is inscribed in a square. Find the area of the shaded region. (Use $\pi = 3.14$)

Solution : Area of the circle $= \pi r^2$

$= 3.14 \times (7.5)^2$ cm^2

$= 176.625$ cm^2

Clearly, side of the square = diameter of the circle = 15 cm

So, area of the square $= 15^2$cm^2 = 225 cm^2

Therefore, area of the shaded region

$= 225$ cm^2 $- 176.625$ cm^2 $= 48.375$ cm^2

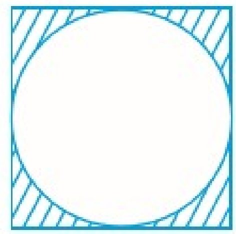
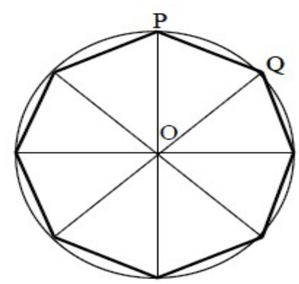


Fig.11.4

Activity

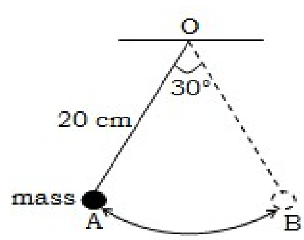
1. The diagram below shows a regular octagon inscribed in a circle of radius 4 cm at centre O. [diagram is not drawn to scale]



Calculate the following:

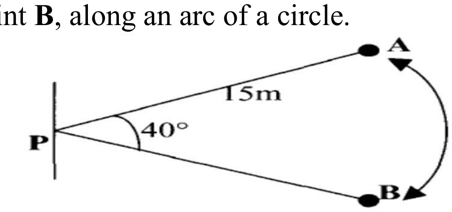
(a) length of the arc
(b) area of sector OPQ
(c) area of triangle OPQ
(d) area of the octagon

2. A simple pendulum consists of a 20 cm long string with a mass attached at one end and the other end fixed to point O. When released from point A, the mass swings to point B along an arc of a circle.



Calculate the area of the region swept by the string when the string moves from point A to point B.

3. A goat is tied to a pole at point **P** by a rope that is 15m long. With the rope stretched tight, the goat moves from point **A** to point **B**, along an arc of a circle.



(a) What distance does the goat travel when it moves from point **A** to point **B**?
(b) Calculate the area of region swept by the rope, when the goat moves from point **A** to **B**?

Strand 5: Trigonometry **Sub Strand 5.1: Non-Right Angled Triangles**

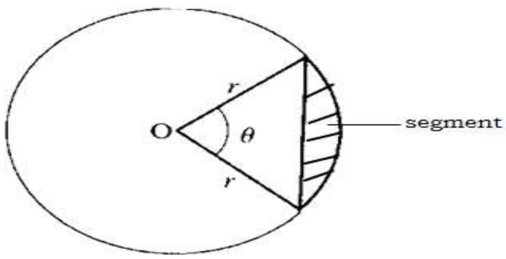
Lesson 58: Circular Measure

Learning outcome: Find the area of segment using the formula.

Area of Segment

Segment is the area bounded by a chord and an arc.

Area of Segment = Area of the Sector – Area of the triangle

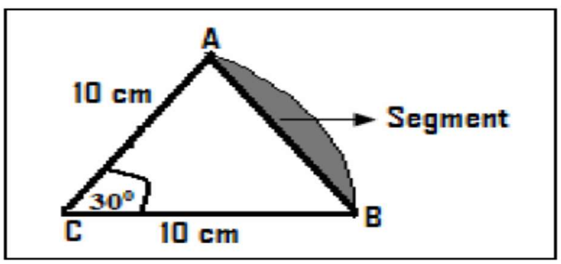


$$A_{segment} = \frac{1}{2}r^2(\theta - \sin\theta)$$

θ in radians; θ in Sin θ in degrees

Example 1

Find the area of the shaded region.



Step 1: Find the area of the sector

$$Area_{sector} = \frac{1}{2}r^2\theta$$

$$\theta = 30^0 \Rightarrow \frac{\pi}{180}\times 30 \Rightarrow \theta = 0\cdot 524$$

$$A = \frac{1}{2}(10)^2(0\cdot 524) \rightarrow A \approx 26\cdot 2\,cm^2$$

Step 2: Find the area of the triangle.

$$Area\ of\ \Delta = \frac{1}{2}ab\,Sin\,C$$

$$Area\ of\ \Delta = \frac{1}{2}(10)(10)Sin\,30^o$$

$$Area\ of\ \Delta = 25\,cm^2$$

Step 3: Find the area of the shaded region.

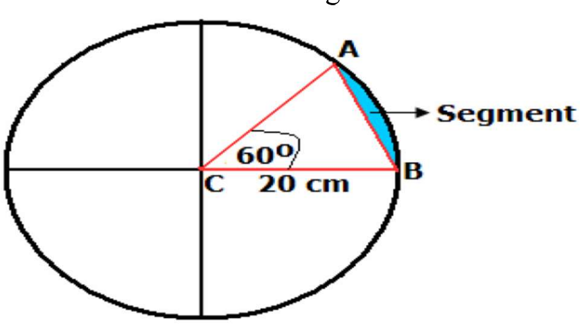
$$Area\ of\ segment = Area\ of\ sector - Area\ of\ \Delta$$

$$Area\ of\ segment = 26\cdot 18\,cm^2 - 25\,cm^2$$

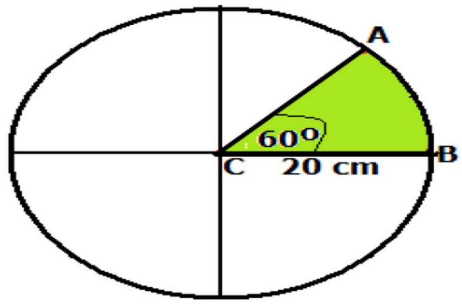
$$Area\ of\ segment = 1\cdot 18\,cm^2$$

Example 2

Find the area of the shaded segment.



First find the area of the sector

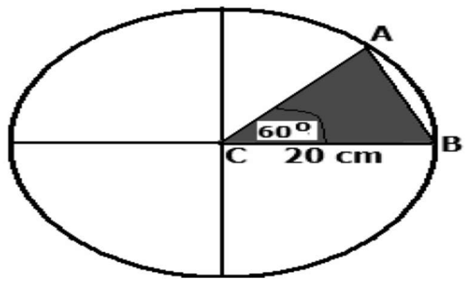


$$A = \frac{1}{2}R^2\theta$$

$$\theta = \frac{60}{1} \times \frac{\pi}{180} \rightarrow \frac{60\pi}{180} \rightarrow \frac{\pi}{3}\,radian$$

$$A = \frac{1}{2}(20)^2\left(\frac{\pi}{3}\right) \rightarrow A = 209\cdot 4395\,cm^2$$

Now find the area of the triangle ABC



$$Area = \frac{1}{2}ab\sin C$$

$$Area = \frac{1}{2}(20cm)(20cm)\sin 60^0$$

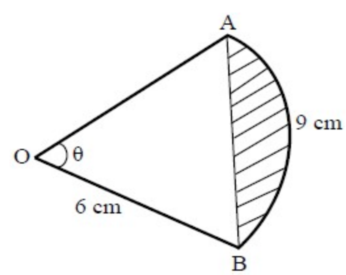
$$Area = 173\cdot 2050808cm^2$$

Area of segment = Area of sector – Area of triangle

$$209\cdot 4395102\,cm^2 - 173\cdot 2050808cm^2 \approx 36\cdot 23\,cm^2$$

<u>Activity</u>

1. The diagram below shows a sector of a circle with the radius of 6 cm and the length of the arc is 9 cm.

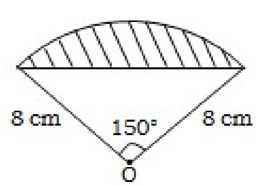


   (a) Show that the angle θ = 1.5 rad.

   (b) Calculate the area of the sector OAB.

   (c) Calculate the area of the shaded segment.

2. The sector shown in the diagram below is part of a circle with radius 8 cm and centre O.



Calculate the area of the shaded region

Strand 5: Trigonometry **Sub Strand 5.1: Non-Right Angled Triangles**

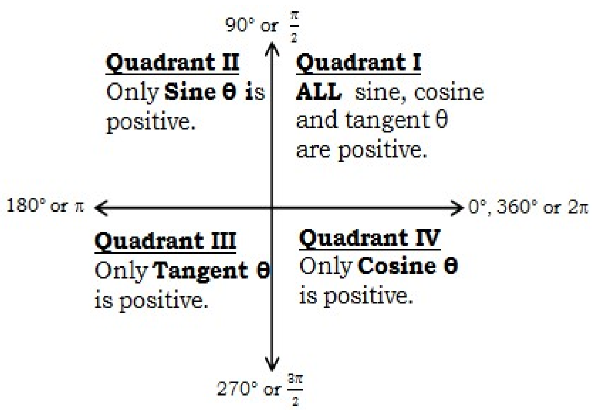
Lesson 59: Solving Trigonometric Equations

Learning outcome: Solve trigonometry equation using the quadrant rules.

When solving any trigonometric equation, emphasis must be given to the angle, **θ**, which can be either in **degrees or radians**.

To **solve for θ**, follow solving an algebraic equation:

- The value consisting θ, to be removed last.
- Do opposite operation on both sides of the equation till you reach the trigonometric expression containing sine, cosine or tangent.
- At this point in time, keep in mind that there will be at least two angles, within 0˚ to 360˚ or 2π radians.
- If Trig expression is positive, then you will directly get the acute angle θ_1. If Trig expression is negative, then ignore the negative sign to get the acute angle and use this to find the angle θ_1.
- Use **quadrants** to find the other angle θ_2. Angles will be considered from the positive x – axis.



	Fourth Quadrant	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
Angle	$-90° < \theta < 0°$ $-\frac{\pi}{2} < \theta < 0$	$0° < \theta < 90°$ $0 < \theta < \frac{\pi}{2}$	$90° < \theta < 180°$ $\frac{\pi}{2} < \theta < \pi$	$180° < \theta < 270°$ $\pi < \theta < \frac{3\pi}{2}$	$270° < \theta < 360°$ $\frac{3\pi}{2} < \theta < 2\pi$
Reference Angle	$-\theta$ $-\theta$	θ θ	$180° - \theta$ $\pi - \theta$	$\theta - 180°$ $\theta - \pi$	$360° - \theta$ $2\pi - \theta$

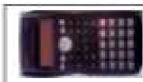
Example 1: Solve $\tan\theta - 1 = 0,\ 0 \le \theta \le 2\pi$

Last

$\tan\theta - 1 = 0,$ $0 \le \theta \le 2\pi$ means that angle to be between $0 - 2\pi$

$\tan\theta - 1 + 1 = 0 + 1,$
$\tan\theta = 1$
$\theta_1 = \tan^{-1} 1$
$\theta_1 = 45°$

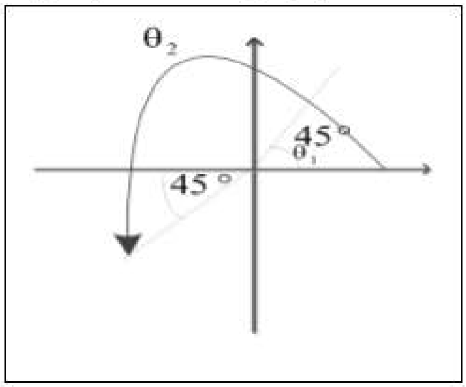
We reached at the trig expression: Consider the two quadrants. Find the acute angle in Q I. Note that calculator Mode to be in degrees.


Press | Shift | tan | 1 | =

- Use quadrants to find the angle θ_2. Consider sign (+) of 'tan', that is in Q III

$\theta_2 = 180 + \theta_1$
$= 180 + 45$
$= 225°$

$\theta = 45^o, 225^o$ or
$\theta \in \{45^o, 225^o\}$



Example 2: Find the solution set for $2Cos\theta + \sqrt{3} = 0, 0^o \le \theta \le 360^0$

$2Cos\theta + \sqrt{3} = 0,$ $0^o \le \theta \le 360^0$ Means that angle to be between 0 – 360°

Last

$2Cos\theta + \sqrt{3} - \sqrt{3} = 0 - \sqrt{3}$

$\frac{2Cos\theta}{2} = \frac{-\sqrt{3}}{2}$

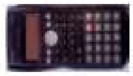
$Cos\theta = \frac{-\sqrt{3}}{2}$

We reached at the trig expression: Consider the two quadrants. But before that, find the acute angle by ignoring the negative sign (–). Note that calculator Mode to be in degrees.

- **Acute angle:**

$$\theta = Cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^o$$

$$\alpha = 30^o$$



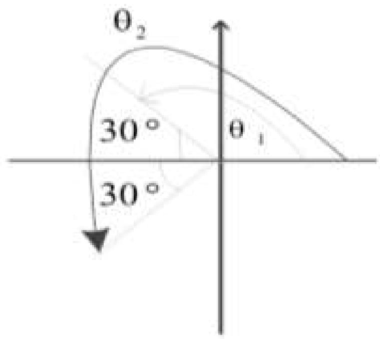
When dealing with surds, press the division sign (÷), that is Press

[Shift] [cos] [(] [√3] [÷] [2] [)] [=]

- Use quadrants to find the angles θ_1 and θ_2. Consider negative sign (−) of Cos, that is in Q II / III

$$\theta_1 = 180 - 30 = 150^o$$

$$\theta_2 = 180 + 30 = 210^o$$



$$\theta = 150^o, 210^o \; or \quad \theta \in \{150^o, 210^o\}$$

EXAMPLE 3: Solve the trigonometric equation $\sin(x+30^0) = 0.4$, where $-180^0 \le x \le 180^0$.

Acute angle:

$$\sin(x+30^0) = 0.4$$

$$(x+30^0) = \sin^{-1} 0.4$$

$$\alpha = 23.58^o$$

It already has trig expression: Consider the two quadrants. Note that calculator Mode to be in degrees.

Press [Shift] [sin] [0] [.] [4] [=]

- Use quadrants to find the angles θ_1 and θ_2. Consider positive sign (+) of sin, that is in Q I / II

$$\theta_2 = 180 - \alpha = 180 - 23.58 = 156.42^o \qquad \alpha = \theta_1 = 23.58^o$$

$QI \,\&\, II$:

$QI : x + 30^0 = 23.58,$

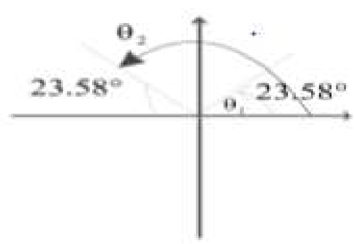
$x + 30^0 = 23.58,$

$x = -6.42^o$

$QII : x + 30^0 = 156.42$

$x = 126.42^o$

$\therefore x \in \{-6.42^o, 126.42^o\}$



Example 4

Find the solution set of the equation $7 \tan \theta = 2\sqrt{3} + \tan \theta$ in the interval $0° \leq \theta < 360°$.

Solution *How to Proceed*

(1) Solve the equation for $\tan \theta$:

$$7 \tan \theta = 2\sqrt{3} + \tan \theta$$
$$6 \tan \theta = 2\sqrt{3}$$
$$\tan \theta = \frac{\sqrt{3}}{3}$$

(2) Since $\tan \theta$ is positive, θ_1 can be a first-quadrant angle:

$$\theta_1 = 30°$$

(3) Since θ is a first-quadrant angle, $R = \theta$:

$$R = 30°$$

(4) Tangent is also positive in the third quadrant. Therefore, there is a third-quadrant angle such that $\tan \theta = \frac{\sqrt{3}}{3}$. In the third quadrant, $\theta_2 = 180° + R$:

$$\theta_2 = 180° + R$$
$$\theta_2 = 180° + 30° = 210°$$

Answer The solution set is {30°, 210°}.

Class Activity

Solve the following trigonometric equations.

1. $2\,\text{Cos}\,\theta + \sqrt{3} = 0$ for $0^\circ \leq \theta \leq 360^\circ$

2. $2\,\text{Sin}\left(x + \frac{\pi}{4}\right) = 1$ where $0 \leq x \leq 2\pi$

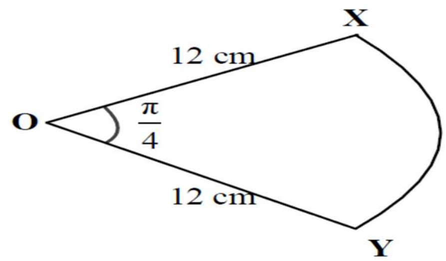
3. $2\,\text{Cos}\;2x = \sqrt{3}$ for $0 \leq x \leq 2\pi$

RATU NAVULA COLLEGE

YEAR 12 MATHEMATICS - WORKSHEET 4

Strand 5: Trigonometry **Sub Strand 5.1: Non-Right Angled Triangles**

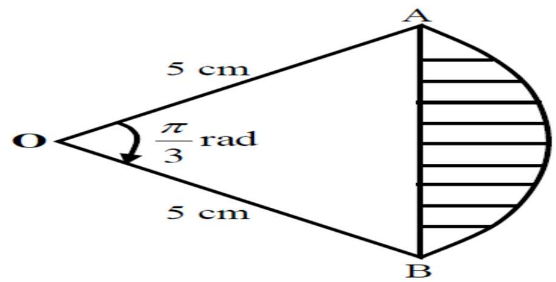
1. A sector with centre O, radius 12 cm and angle $\frac{\pi}{4}$ radians is show.



   (a) Calculate the length of arc XY.

   (b) Calculate the area of the sector.

2. The diagram below shows a sector of a circle with the radius of 5 cm and the angle of $\frac{\pi}{3}$ radians. Point O is the centre of the circle.

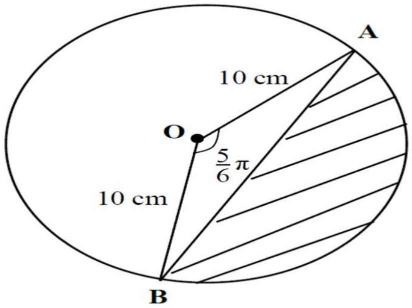


   (a) Calculate the area of the sector OAB.

   (b) Calculate the area of the triangle OAB.

   (c) Calculate the area of the shaded segment.

3. The diagram below shows a circle with centre O and radius 10 cm. The arc AB forms an angle of $\frac{5}{6}\pi$ radians from the centre.



   Calculate the **area** of the shaded segment.

4. Solve the trigonometric equation $\mathbf{2\cos\theta = 1}$ for $\mathbf{0 \le \theta \le 360°}$

5. Solve $\mathbf{2\ \cos x = \sqrt{3}}$ for $\mathbf{0 \le x \le 2\pi}$.